Problem Set 3 - ECMA 31130

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Part 1 (25 points)

We consider a measurement system for a random variable of interest X which is unobserved to the econometrician. Instead we observe two variables Y_1, Y_2 related to X in the following way:

$$Y_{i1} = X_i + \epsilon_{i1}$$
$$Y_{i2} = X_i + \epsilon_{i2}$$

and we assume that $\mathbb{E}(X) = \mu$, $\text{Var}(X) = \sigma_x^2$, $X \perp \epsilon_1 \perp \epsilon_2$, $\mathbb{E}(\epsilon_1) = \mathbb{E}(\epsilon_2) = 0$ and $\text{Var}(\epsilon_1) = \sigma_1^2, \text{Var}(\epsilon_2) = \sigma_2^2$. Given a set of independent draws $\big(Y_{i1}, Y_{i2})_{i=1..n}$ from our model, we consider the following estimator for μ:

$$\mu_n \;=\; \alpha \times \frac{1}{n}\sum_{i=1}^{n} Y_{1i} + (1-\alpha) \times \frac{1}{n}\sum_{i=1}^{n} Y_{2i}$$

1. (5 points) Show that μ_n is an unbiased estimator of μ.
2. (5 points) Show that μ_n is a consistent estimator of μ.
3. (5 points) Compute the variance of μ_n as a function of α and the other parameters.
4. (6 points) Find α as a function of σ_1, σ_2 that minimizes the variance. Interpret the results.
5. (4 points) We assumed through this question that $\epsilon_1, \epsilon_2, X$ were independent. Could we find a weaker assumption and keep the unbiasedness?

Part 2 (25 points)

We keep the same model as in Part 1: $Y_{ik} = X + \epsilon_{ik}$ with $\epsilon_1 \perp \epsilon_2$, $\mathbb{E}(\epsilon_k) = 0$ and $\text{Var}(\epsilon_k) = \sigma_k^2$. In this part 2, **we further assume** that the distributions of ϵ_k are normal $\epsilon_k \sim \mathcal{N}(0, \sigma_k^2)$, and that $\text{Var}(X) = 0$ such that $X = \mu$ is a constant. Under these assumptions Y_1, Y_2 are normally distributed as well.

1. (5 points) Show that under the current assumptions ($X = \mu$, $\sigma_x = 0$) we have that $Y_1 \perp Y_2$. If, as in Part 1, we had that $\sigma_x > 0$ what would $\text{Cov}(Y_1, Y_2)$ be?

2. (8 points) Given that Y_1, Y_2 are independent here and the underlying distributional assumption, compute the likelihood and the log-likelihood of observing $\left(Y_{i1}, Y_{i2}\right)_{i=1}^{n}$ given parameters μ, σ_1, σ_2.

3. (7 points) Finally, solve for the maximum likelihood estimator for μ (consider here that the variances σ_1, σ_2 are known). Explain your steps.

4. (5 points) Compare and contrast this estimator to the optimal weighting found in question (3), part 1.

Part 3 - True, False or Uncertain (25 points)

For each of the following questions, please answer whether the statement is True, False or Uncertain, and then justify your answer with a short paragraph, a proof, or a counterexample as appropriate. No credit will be given for answers without any justification.

1. (5 points) Let U be a uniform random variable on $[0, 1]$, then $\mathbb{E}[U^\gamma] = \frac{1}{\gamma}$.

2. (5 points) Consider the ML estimator θ_n for a parameter θ. This estimator is always unbiased for θ.

3. (5 points) All econometric estimators achieve $\sqrt{n}$ convergence to a normal distribution, i.e.

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{\text{d}} \mathcal{N}(0, \sigma^2)$$

4. (5 points) Multinomial logit and conditional logit models yield solutions over the set of options such that the odds of preferring one option over another can change when a new alternative is introduced.

5. (5 points) Let $X_1, \ldots, X_n$ denote the number of Bernoulli trials with p until the first success happens, which can be modeled as a geometric distribution parameterized by p. Therefore, if k is the number of trials, then

$$\Pr\{X_i = k\} \ = \ (1 - p)^{k-1} p$$

This implies that $\mathbb{E}[X_i] = \frac{1}{p}$.

Part 4 - A DIY ML Estimator (25 points)

You are asked to construct the ML algorithm from scratch for a simple model. Follow the sequence of steps provided to get full credit for this question. DO NOT use any pre-packaged, high-level econometric functions to work through this question, as your grade will reflect your understanding and execution of the ML algorithm without the aid of Stata-like commands. This will prove useful later in your career when you develop and estimate structural models!

1. Simulate 1000 independent draws (indexed by i) of an unobserved variable ϵ_i drawn from $\mathcal{N}(0,4)$. Similarly, generate data for the observed variable X_i from $\mathcal{N}(0,1)$. Finally, construct

$$Y_i \;=\; X_i\beta + \epsilon_i$$

   Take $\beta = 1.5$. You can delete ϵ_i from your data now, to replicate the conditions under which you would normally perform such regressions.

Now, we treat β and σ_ϵ^2 as the parameters of interest, which can be recovered using MLE from the data with observations for Y_i, X_i.

2. Construct a function that computes the log-likelihood associated with the entire sample, taking inputs for values of β and σ_ϵ^2.

3. Evaluate and plot the log-likelihood as a function of β, fixing $\sigma_\epsilon^2 = 4$. Evaluate and plot the log-likelihood as a function of σ_ϵ^2, fixing $\beta = 1.5$. Comment on the identification of the model parameters using ML using the plots generated.

4. Using the function above, attempt to maximize the log-likelihood (or minimize the negative log-likelihood) to recover the ML estimates for β and σ_ϵ^2. You may use an optimizer function or package.

5. Compute the standard errors for your estimators using the Central Limit result for the ML estimator.

6. Compute the standard errors for your estimators using the bootstrap method.